Another Hilbert inequality and critically separated interpolation nodes

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Abstract: Estimates on the condition number of Vandermonde matrices have implications on several algorithms ranging from polynomial interpolation to sparse super resolution in fluorescence microscopy. Classically, the situation is studied for monomials on real intervals, the complex unit disk, and the complex unit circle. Except for roots of unity and well separated nodes on the unit circle, the condition number grows strongly with increasing polynomial degree. Here, we show that the condition number of the Vandermonde matrix for a particular instance of critically separated nodes on the complex unit circle grows logarithmically with the polynomial degree. The proof is based on a variant of Hilbert's inequality with remainder term.

1 Introduction and known results

For $m, n \in \mathbb{N}$, $z_j \in \mathbb{C}$, $j = 1, \dots, m$, we define the Vandermonde matrix

$$A := A(n, \{z_j\}) := (z_j^{k-1})_{j,k=1}^{m,n} \in \mathbb{C}^{m \times n}$$

A classical result [2, Thm. 4.1] shows

$$\operatorname{cond} A \geq \sqrt{\frac{2}{n+1}} \cdot \left(1 + \sqrt{2}\right)^{n-1}$$

for $m = n$ and arbitrary real nodes $z_j \in [-1, 1]$, where we note in passing that we consider the basis of monomials only and other bases like Chebyshev polynomials overcome this ill-conditioning behaviour. Similar results are available for Vandermonde matrices with nodes in the complex unit disk, cf. [1, Thm. 5]. The situation changes dramatically, if we consider nodes on the complex unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Here the primal example is of course the Fourier matrix $F := (\mathrm{e}^{2\pi \mathrm{i} kj/n})_{j,k=0,\dots,n} \in \mathbb{C}^{n \times n}$ which is unitary up to normalization and thus has condition number 1. More general, nodes on the unit circle $z_j = \mathrm{e}^{2\pi \mathrm{i} t_j}$, $t_j \in [0, 1)$, are called *well separated* if

$$nq > 1$$

where $q := \min_{j \neq \ell} \min_{r \in \mathbb{Z}} |t_j - t_\ell + r|$. An approach via extremal functions [5] or via a generalized Hilbert inequality [1] then yields

$$\operatorname{cond} A \leq \sqrt{\frac{nq+1}{nq-1}}.$$

Obviously, this bound detoriates for $nq \to 1$ although full row rank m is still guaranteed via a Vandermonde determinant argument since $m \leq 1/q \leq n$. Moreover, for $nq < 1$ and n large enough, there can be placed $m > n$ equispaced nodes on the unit cirlce and thus the matrix A cannot have rank m.

2 Specific setup and main result

In the following we discuss one particular critically separated case $nq = 1$ and show that unlike for the Fourier matrix, the condition number grows with n.

Fig. 1: Nodes with some larger gap around $\frac{1}{2}$ and $0 (= 1 \mod 1)$

Let $n \in \mathbb{N}$, $q = \frac{1}{2n+1}$, $m = 2n$, and nodes be given by

$$x_j = \begin{cases} jq, & j = 1, \dots, n, \\ \left(j + \frac{1}{2}\right) q, & j = n+1, \dots, 2n, \end{cases}$$

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see Figure 1 for an illustration. Now, we define the Vandermonde matrix

$$A_n := \left(\mathrm{e}^{2\pi \mathrm{i} k x_j}\right)_{j=1,\ldots,2n;k=-n,\ldots,n} \in \mathbb{C}^{2n\times 2n+1}$$

and the symmetric Toeplitz matrix

$$T_n := \frac{1}{2n+1}\left(\frac{1}{\cos\left(\frac{\pi(j-\ell)}{2n+1}\right)}\right)_{j,\ell=1,\ldots,n} \in \mathbb{R}^{n\times n}.$$

The following two lemmata discuss the close relationship of these matrices and approximate eigenvectors of the matrix T_n.

Lemma 2.1 *With the above notation, the extremal singular values of A_n and the maximal eigenvalue of T_n are related via*

$$\sigma_{\min}^2 A_n = 1 - \lambda_{\max} T_n, \qquad \sigma_{\max}^2 A_n = 1 + \lambda_{\max} T_n.$$

Proof. We have

$$K_n := \frac{1}{2n+1} A_n A_n^* = \frac{1}{2n+1}\left(\frac{\sin(2n+1)\pi(x_j - x_\ell)}{\sin \pi(x_j - x_\ell)}\right)_{j,\ell}$$
$$= \begin{pmatrix} I_n & T_n' \\ T_n' & I_n \end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \quad (T_n')_{j,\ell} := (-1)^j (T_n)_{j,\ell}(-1)^\ell$$

and since the A_n has full rank $2n$, the matrix K_n is symmetric positive definite. A theorem of Jordan-Wielandt [4, Thm. 7.3.3] yields the result. □

Lemma 2.2 *There exist constants $C_1, C_2 > 0$ such that for sufficiently large $n \in \mathbb{N}$, the vectors $\ell, v, w \in \mathbb{R}^n$,*

$$v_j := \frac{1}{\sqrt{\sin\frac{2\pi j}{2n+1}}}, \qquad \ell_j := \log\left(\min\{j, n+1-j\}\right), \qquad w_j := \frac{1}{2}\ell_j \cdot (v_j + v_{n+1-j}), \qquad j = 1,\ldots,n,$$

fulfil

$$w^\top w \geq C_1 n \log^3 n \qquad \textit{and} \qquad w^\top (I_n - T_n) w \leq C_2 n \log n.$$

Proof. Please note that the non-numbered constant C may differ at each appearance. Only considering the first half of the vector and estimating $\sin x \leq x$, we directly compute

$$w^\top w \geq (2n+1)\sum_{j=1}^{\lfloor n/2\rfloor} \frac{\log^2 j}{2\pi j} \geq C_1 n \log^3 n.$$

Subsequently, we bound $w^\top (I_n - T_n) w$ from above, this involves several preparatory steps: The definition of the vector v is motivated by the continuous analogue

$$\frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\cos(x-y)}\cdot\frac{1}{\sqrt{\sin 2x}}\mathrm{d}x = \frac{1}{\pi}\int_{-1}^{1}\frac{1}{1+ts}\cdot\frac{1}{\sqrt{1-t^2}}\mathrm{d}t = \frac{1}{\sqrt{1-s^2}} = \frac{1}{\sqrt{\sin 2y}},$$

where the first equality uses $t = \tan x$ and $s = \tan y$ and the second equality follows from $t = \frac{1}{2}\left(z + z^{-1}\right)$ and Cauchy's integral formula. For fixed $y \in (0, \frac{\pi}{2})$, the integrand $S_y : (0, \frac{\pi}{2}) \to \mathbb{R}$,

$$S_y(x) = \frac{1}{\cos(x-y)}\cdot\frac{1}{\sqrt{\sin 2x}},$$

is convex and with $h = \pi/(2n+1)$ and $j = 1,\ldots,n$, we obtain

$$\underbrace{\frac{1}{\pi}\int_0^{\pi/2} S_{jh}(x)dx}_{v_j} > \frac{1}{\pi}\int_{h/2}^{\pi/2} S_{jh}(x)\mathrm{d}x > \underbrace{\frac{1}{2n+1}\sum_{i=1}^{n} S_{jh}\left(\frac{i}{2n+1}\right)}_{(T_n v)_j} > \frac{1}{\pi}\int_h^{\pi/2-h/2} S_{jh}(x)\mathrm{d}x$$

where the last inequality follows by leaving out the minimal summand and moving the remaining ones inwards. Together with

$$\int_0^h S_{jh}(x)\mathrm{d}x = \int_0^h \frac{1}{\cos(x-jh)}\cdot\frac{1}{\sqrt{\sin 2x}}\mathrm{d}x \leq \max_{x\in[0,h]}\frac{1}{\cos(jh-x)}\int_0^h \frac{1}{\sqrt{4x/\pi}}\mathrm{d}x \leq \frac{C\sqrt{n}}{j}$$

and an analogous estimate for the integral on the interval $[\pi/2 - h/2, \pi/2]$, this yields

$$(T_n\overline{v})_j < \overline{v}_j \le (T_n\overline{v})_j + C\sqrt{n}\max\left\{\frac{1}{j}, \frac{1}{n+1-j}\right\}$$

for the symmetrized vector $\overline{v} \in \mathbb{R}^n$, $\overline{v}_j = \frac{1}{2}(v_j + v_{n+1-j})$. Together with $\sin x \ge 2x/\pi$, $L := \operatorname{diag}(\ell)$, and using the symmetry of the vectors $w = L\overline{v}$ and $\overline{v}$, we get

$$w^\top L(\overline{v} - T_n\overline{v}) \le C\sqrt{n}\sum_{j=1}^{\lceil n/2\rceil}\ell_j^2\overline{v}_j\frac{1}{j} \le Cn\sum_{j=1}^{\lceil n/2\rceil}\frac{\log^2 j}{j^{3/2}} \le Cn.$$

Secondly, there exists $\kappa \in (0,1)$ such that

$$\frac{1}{2n+1}\sum_{i=1}^{n}\frac{(\ell_j - \ell_i)\overline{v}_i}{\cos\frac{\pi(j-i)}{2n+1}} \le \begin{cases} C\sqrt{\frac{n}{j}}, & \kappa n/2 < j \le \lceil n/2\rceil, \\ 0, & j \le \kappa n/2. \end{cases} \tag{1}$$

To see this, we start by noticing the symmetry of ℓ and $\overline{v}$ as well as $\cos x \ge \cos(\pi/2 - x)$, $0 \le x \le \pi/4$, such that

$$\frac{(\ell_j - \ell_i)\overline{v}_i}{\cos\frac{\pi(j-i)}{2n+1}} < \frac{(\ell_j - \ell_i)\overline{v}_i}{\cos\frac{\pi(j-(n+1-i))}{2n+1}} < \frac{(\ell_j - \ell_i)\overline{v}_i}{\sin\frac{\pi(j+i)}{2n+1}}, \quad \frac{(\ell_j - \ell_{n+1-i})\overline{v}_{n+1-i}}{\cos\frac{\pi(j-(n+1-i))}{2n+1}} = \frac{(\ell_j - \ell_i)\overline{v}_i}{\cos\frac{\pi(j-(n+1-i))}{2n+1}} < \frac{(\ell_j - \ell_i)\overline{v}_i}{\sin\frac{\pi(j+i)}{2n+1}}.$$

for $i, j < n/2$. We get for $j \le \lceil n/2\rceil$ the bound

$$\frac{1}{2n+1}\sum_{i=1}^{j}\frac{(\ell_j - \ell_i)\overline{v}_i}{\sin\frac{\pi(j+i)}{2n+1}} \le C\sum_{i=1}^{j}\frac{\log(j/i)\overline{v}_i}{j+i} \le C\frac{1}{j}\sum_{i=1}^{j}\frac{\log(j/i)}{\sqrt{i/n}} \le C\sqrt{\frac{n}{j}}\int_0^1\frac{\log(1/x)}{\sqrt{x}}\mathrm{d}x \le C\sqrt{\frac{n}{j}}.$$

Moreover, the subsequent summands are all negative and can be estimated by

$$\begin{aligned}
\frac{1}{2n+1}\sum_{i=j+1}^{\lceil n/2\rceil}\frac{(\ell_j - \ell_i)\overline{v}_i}{\sin\frac{\pi(j+i)}{2n+1}} &\le \frac{1}{2n+1}\sum_{i=\lceil n/4\rceil+1}^{\lceil n/2\rceil}\frac{(\log j - \log i)\overline{v}_i}{\sin\frac{\pi(j+i)}{2n+1}} \\
&\le \frac{1}{2n+1}\sum_{i=\lceil n/4\rceil+1}^{\lceil n/2\rceil}\frac{(\log j - \log\lceil n/4\rceil)\overline{v}_i}{\sin\frac{\pi(j+i)}{2n+1}} \\
&\le C\log(4j/n)\frac{1}{2n+1}\sum_{i=\lceil n/4\rceil+1}^{\lceil n/2\rceil}\frac{\overline{v}_i}{\sin\frac{\pi(j+i)}{2n+1}} \\
&\le C\log(4j/n)\int_{1/4}^{1/2}\frac{\mathrm{d}x}{\sqrt{x}\left(\frac{j}{n} + x\right)} \\
&\le C\log(4j/n)\sqrt{\frac{n}{j}},
\end{aligned}$$

where the first inequality either adds positive or neglects negative terms, respectively. Combining both estimates leads to

$$\frac{1}{2n+1}\sum_{i=1}^{n}\frac{(\ell_j - \ell_i)\overline{v}_i}{\cos\frac{\pi(j-i)}{2n+1}} \le \frac{2}{2n+1}\sum_{i=1}^{\lceil n/2\rceil}\frac{(\ell_j - \ell_i)\overline{v}_i}{\sin\frac{\pi(j+i)}{2n+1}} \le C\left(1 + \log(4j/n)\right)\sqrt{\frac{n}{j}}$$

which is non-positive for $\log(4j/n) \le -1$ and thus shows the inequality (1). Together with $\sin x \ge 2x/\pi$ and again the symmetry of w and $\overline{v}$ we get

$$\begin{aligned}
w^\top(LT_n\overline{v} - T_n w) &= \sum_{j=1}^{n} w_j\left(T_n(\ell_j\overline{v} - w)\right)_j \\
&\le \frac{C}{n}\sum_{j=1}^{\lceil n/2\rceil} w_j\sum_{i=1}^{n}\frac{(\ell_j - \ell_i)\overline{v}_i}{\cos\frac{\pi(j-i)}{2n+1}} \\
&\le C\sum_{j=\lceil\kappa n/2\rceil}^{\lceil n/2\rceil}\frac{\sqrt{n}\log j}{\sqrt{j}}\sqrt{\frac{n}{j}} \\
&\le Cn\log n
\end{aligned}$$

Altogether, the second result follows from $w^\top(I_n - T_n)w = w^\top L(\overline{v} - T_n\overline{v}) + w^\top(LT_n\overline{v} - T_n w)$. □

Theorem 2.3 *There exist a constant* $C_0 > 0$ *such that for sufficiently large* $n \in \mathbb{N}$ *and with the above notation, we have*

$$1 - \frac{1}{C_0^2 \log^2 n} \leq \|T_n\| < 1.$$

and in particular

$$\operatorname{cond} A_n \geq C_0 \log n.$$

Proof. The upper bound follows from Lemma 2.1 and symmetry of T_n by $0 < \sigma_{\min}^2 A_n = 1 - \lambda_{\max} T_n = 1 - \|T_n\|$. The lower bound follows from Lemma 2.2 by

$$\|T_n\| \geq \frac{w^\top T_n w}{w^\top w} \geq 1 - \frac{w^\top (I_n - T_n) w}{w^\top w}.$$

□

Remark 2.4 The vector v in Lemma 2.2 would result in a weaker final result by

$$v^\top v \geq Cn \log n, \quad v^\top (I - T_n) v \leq Cn, \quad \text{and} \quad \operatorname{cond} A_n \geq C\sqrt{\log n}.$$

Finally, we would like to mention the following similar result for the Hilbert matrix. We have

$$1 - \frac{\pi^4}{2\log^2 n} + O\left(\frac{\log\log n}{\log^3 n}\right) \leq \|H_n\| < 1, \qquad H_n := \frac{1}{\pi}\left(\frac{1}{j+\ell}\right)_{j,\ell=1,\dots,n} \in \mathbb{R}^{n\times n},$$

where the upper bound might be seen as a variant of the famous Hilbert inequality and the lower bound can be found in [3]. While we have the very same rate for the second term, our constant certainly is an artefact of our proof technique. Unfortunately, we were not able to relate T_n and H_n directly. The numerical test in Figure 2 suggests a "true" constant $C = 4/\pi^2$ in Theorem 2.3.

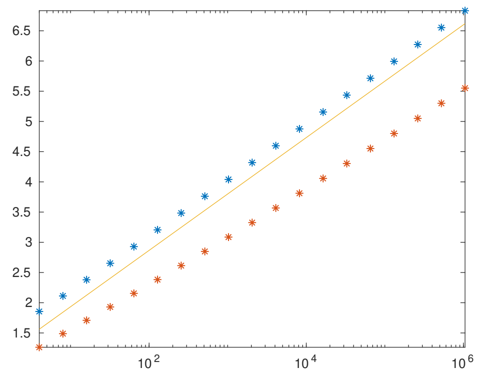

Fig. 2: Condition number of the Vandermonde matrix A_n with respect to n (blue stars), numerical value of $\sqrt{1 - w^\top w / w^\top T w}$ for the test vector w (red stars), and $1 + \frac{4}{\pi^2}\log(n)$ (yellow line). Note that the norm of the matrix T_n can be computed explicitly for moderate n and via Matlab's routine `svds`, where we used a fast matrix vector multiplication via fast Fourier transforms, for large n.

Acknowledgements SK thanks Albrecht Böttcher for early discussions on the subject. SK gratefully acknowledges support by the DFG within the Collaborative Research Center 944: "Physiology and dynamics of cellular microcompartments" and by the Volkswagen Foundation project "Stability of Moment Problems and Super-Resolution Imaging".

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