7: Gaussian Fluctuations in Models of Statistical Mechanics – Fine Asymptotics for the Magnetization

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SPP2265 Random Geometric Systems

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Possible connections to projects: 4, 16, 23, 31, 34

Magnetization

Consider random spins $\sigma_1, \dots, \sigma_N \in \{-1, 1\}$ w.r.t. a Gibbs measure of the form

$$\mu_{N,\beta}(\sigma) = \frac{1}{Z_{N,\beta}} e^{-\beta H_{N,\beta}(\sigma)}, \quad \beta > 0$$

with a model-dependent Hamiltonian $H_{N,\beta}$.

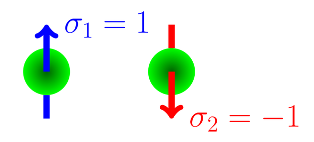


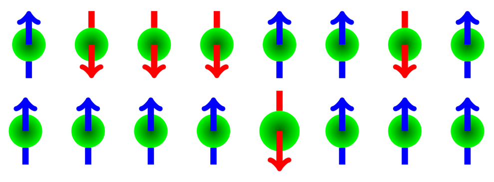
Fig. 1: A spin config. (σ_1, σ_2)

The **magnetization** is the average spin $m_N := m_N(\sigma) = \frac{1}{N}\sum_{i=1}^N \sigma_i \in [-1, 1]$.

Curie-Weiss Model – known results

Curie-Weiss model

$$H_N^{(CW)}(\sigma) := -\frac{1}{2N}\sum_{i,j=1}^N \sigma_i\sigma_j = -\frac{N}{2}(m_N(\sigma))^2$$



- $\beta \le 1$: m_N concentrates in 0
- $\beta > 1$: m_N concentrates in $m(\beta) \neq 0$ or in $-m(\beta)$

For high temperatures $\beta < 1$:

- **Gaussian fluctuations** : $\sqrt{N} m_N \xrightarrow{d} \mathcal{N}(1, \frac{1}{1-\beta}), \quad N \to \infty.$
- Berry-Esseen bounds by Stein's method and via mod-Gaussian convergence
- Asymptotics for mixed moments are available

Method of Cumulants

For $j \in \mathbb{N}$, the ***j*-th cumulant** of a real-valued random variable X is given by

$$\kappa_j(X) := (-\mathrm{i})^j \frac{\mathrm{d}^j}{\mathrm{d}t^j} \log \mathbb{E}\left[\mathrm{e}^{\mathrm{i}tX}\right]\Big|_{t=0},$$

if the derivative exists.

The **Statulevičius condition** $\quad |\kappa_j(X)| \le \frac{(j!)^{1+\gamma}}{\Delta^{j-2}} \quad$ for $j \ge 3$ with $\gamma \ge 0$, $\Delta > 0$ implies

- normal approximation with Cramér correction and a rate of convergence in Kolmogorov distance
- mod-Gauss convergence, i.e. $\lim_{n\to\infty} \mathbb{E}\left[\mathrm{e}^{\mathrm{i}tX_n}\right]/\mathbb{E}\left[\mathrm{e}^{\mathrm{i}tZ_n}\right] = \Phi(t)$ for some Φ.

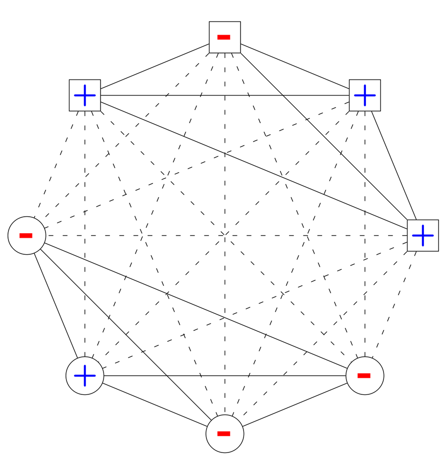
Idea for application: Use known moment expansion.

Block Spin Ising Models

For $\{1, \dots, N\} = S \uplus S^c, |S| = \frac{N}{2}$, N even and $0 \le \alpha \le \beta$:

$$H_{N,\alpha,\beta,S}(\sigma) := -\frac{\beta}{2N}\sum_{i,j \text{ in same block}} \sigma_i\sigma_j - \frac{\alpha}{2N}\sum_{i,j \text{ in diff. blocks}} \sigma_i\sigma_j.$$

For $\mu_{N,\alpha,\beta}(\sigma) := \frac{e^{-H_{N,\alpha,\beta}(\sigma)}}{Z_{N,\alpha,\beta}}$, the vector of block magnetizations $m^N := (\frac{2}{N}\sum_{i\in S}\sigma_i, \frac{2}{N}\sum_{i\notin S}\sigma_i)$ has Gaussian fluctuations for $\alpha + \beta < 2$. Further results for moments are available.



Objectives 1 and Strategies

Derive the Statulevičius condition for the magnetization in the …

1.1 classical Curie-Weiss model via known expansions for the moments

1.2 Ising models with random interactions on an Erdős-Rényi random graph: start with CLT in the annealed setting and derive representations for cumulants

1.3 Block Spin Ising model (2d-vector) with two and more blocks: multivariate cum.

References

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- H. Knöpfel, M. Löwe, K. Schubert, A. Sinulis. *Fluctuation results for general block spin Ising models.* In: J. Stat. Phys. 178.5 (2020), pp. 1175–1200.

Ising-model on Random Graphs – known results

Ising model on a (random) graph $G_N = (\{1, \dots, N\}, E_N)$ with (random) edge set E_N:

$$H_N(\sigma) = -C_N \sum_{(i,j)\in E_N} \sigma_i\sigma_j$$

Beyond the LLN for **Erdős-Rényi random graphs** with edge prob. $p(N)N \to \infty$:

- $\beta < 1$: **Gaussian fluctuations** of $\mathbb{E}_{\mu_{N,\beta}}[\delta_{\sqrt{N}m_N}]$ w.r.t. randomness coming from G_N.
- Berry-Esseen type bounds and concentration results in the quenched and annealed setting via Stein's method for the larger regime $\sqrt{N}p(N) \to \infty$.

In the **classical Ising model** (i.e. E_N is the grid on $[-N, N]^d \cap \mathbb{Z}^d$) for $d \ge 2$

- **Gaussian fluctuations** of magnetization (for $d = 1$ via cumulant bounds)
- for β sufficiently small (or in the presence of an external field): rates of convergence (via cluster expansions and via cumulant bounds/weighted dependency graphs).

Stein's Method

Goal: bound $d(X, Z)$ for $Z \sim \mathcal{N}(0, 1)$ in Wasserstein or Kolmogorov-distance
Let f_h be the solution of **Stein's equation**

$$h(x) - \mathbb{E}h(Z) = f_h'(x) - xf_h(x)$$

for suitable test functions $h \in \mathcal{H}$, then

$$d_{\mathcal{H}}(X, Z) = \sup_{h\in\mathcal{H}}\left|\mathbb{E}h(X) - \mathbb{E}h(Z)\right| = \sup_{h\in\mathcal{H}} \mathbb{E}[f_h'(X) - Xf_h(X)].$$

Ansatz for dependency graphs:
Let $X := \sum_{j=1}^n X_j$, $\mathbb{E}X_i = 0$ and $\mathbb{V}X = \sigma^2$ and let $L = ([n], E)$ be a corresponding dependency graph, i.e. for disconnected sets $A_1, A_2 \subset [n]$ we have $\{X_i : i \in A_1\}$ and $\{X_i : i \in A_2\}$ are independent.
The proof consists of the following steps:

1. $W_i = \sum_{j\notin N_i} X_j$ for $N_i := \{k : k \text{ neighbour of } i \text{ in } L\}$, so $\mathbb{E}X_i f(W_i) = 0$.
2. Taylor expansion $f(X) \approx f(W_i) + (X - W_i)f'(W_i)$.

This yields

$$\mathbb{E}[Xf(X)] = \sum_{i=1}^n \mathbb{E}\big[X_i\big(f(X) - f(W_i)\big)\big] \approx \mathbb{E}\underbrace{\sum_{i=1}^n X_i(X - W_i)}_{=T\approx\sigma^2}\underbrace{f'(W_i)}_{\approx f'(X)} \qquad (1)$$

3. bound $\mathbb{V}(T)$ and $d(X, Z)$ in terms of the maximal degree of the dependency graph.

Weighted Dependency Graphs

A graph $G = (A, E)$ with edge weights $w_e \in [0, 1]$ is called a $(C_1, C_2, \dots)$-weighted dependency graph for a family of random variables $\{Y_\alpha : \alpha \in A\}$ if, for any multiset $B = \{\alpha_1, \dots, \alpha_r\} \subset A$, the following bound on cumulants holds

$$|\kappa(Y_\alpha, \alpha \in B)| \le C_r \max_{T \text{ spanning tree of } G[B]} \text{weight}(T),$$

Idea: For random variables that obey a weighted dependency graph structure there are additional sums in the decomposition of $\mathbb{E}Xf(X)$ in (1), but instead of the maximal degree, one can use the **weighted degree**.

Objectives 2 and Strategies

2.1 Generalize Stein's method to bound the Kolmogorov distance for sums of weakly dependent random variables

2.2 Application of **2.1** to the d-dimensional Ising model

2.3 Consider various applications of weighted dependency graphs for models beyond statistical mechanics, e.g. number of crossings in random pairing.

References

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