Hodge theory for combinatorial geometries

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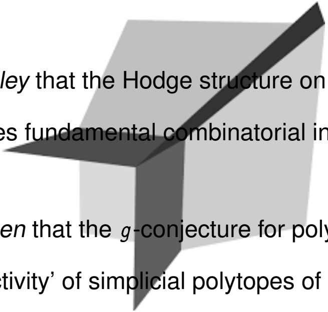
Three ideas:

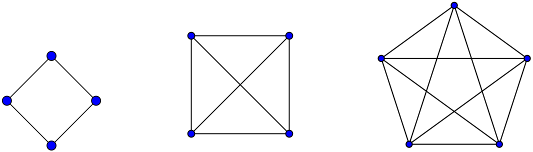
The idea of *Bernd Sturmfels* that a matroid can be viewed as a piecewise linear object, the tropical linear space.



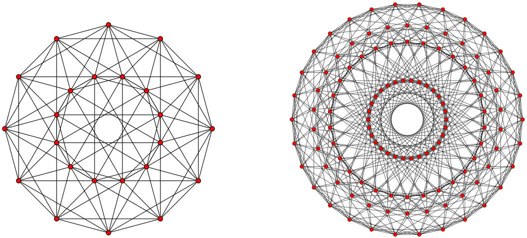
The idea of *Richard Stanley* that the Hodge structure on the cohomology of projective toric varieties produces fundamental combinatorial inequalities.

The idea of *Peter McMullen* that the g-conjecture for polytopes can be proved using the 'flip connectivity' of simplicial polytopes of given dimension.

A *graph* is a 1-dimensional space, with vertices and edges.



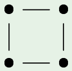
Graphs are the simplest combinatorial structures.



Hassler Whitney (1932): The *chromatic polynomial* of a graph G is the function

$$\chi_G(q) = (\text{the number of proper colorings of } G \text{ with } q \text{ colors}).$$

Example



$$\chi_G(q) = 1q^4 - 4q^3 + 6q^2 - 3q, \qquad \chi_G(2) = 2, \quad \chi_G(3) = 18, \quad \ldots$$

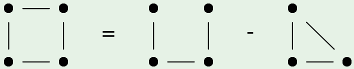
Read-Hoggar conjecture (1968)

The coefficients of the chromatic polynomial $\chi_G(q)$ form a log-concave sequence for any graph G, that is,

$$a_i^2 \geq a_{i-1}a_{i+1} \text{ for all } i.$$

Example

How do we compute the chromatic polynomial? We write



and use

$$\begin{aligned}\chi_{G\setminus e}(q) &= q(q-1)^3\\ \chi_{G/e}(q) &= q(q-1)(q-2).\end{aligned}$$

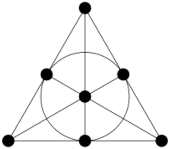
Therefore

$$\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q) = 1q^4 - 4q^3 + 6q^2 - 3q.$$

This algorithmic description of $\chi_G(q)$ makes the prediction of the conjecture interesting.

For any finite set of vectors A in a vector space over a field, define

$$f_i(A) = (\text{number of independent subsets of } A \text{ with size } i).$$



Example

If A is the set of all nonzero vectors in $\mathbb{F}_2^3$, then

$$f_0 = 1, \quad f_1 = 7, \quad f_2 = 21, \quad f_3 = 28.$$

How do we compute $f_i(A)$? We use

$$f_i(A) = f_i(A \setminus v) + f_{i-1}(A \,/\, v).$$

Welsh-Mason conjecture (1969)

The sequence f_i form a log-concave sequence for any finite set of vectors A in any vector space over any field, that is,

$$f_i^2 \geq f_{i-1}\, f_{i+1} \quad \textit{for all } i.$$

Hassler Whitney (1935).

A *matroid* M on a finite set E is a collection of subsets of E, called *independent sets*, which satisfy axioms modeled on the relation of linear independence of vectors.

We write $n+1$ for the *size* of M, the cardinality of the ground set E.

We write $r+1$ for the *rank* of M, the cardinality of any maximal independent set of M.

In all interesting cases, $r < n$.

1. Let G be a finite graph, and E the set of edges.

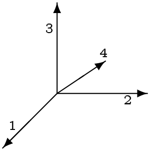
   Call a subset of E independent if it does not contain a circuit.

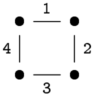
   This defines a *graphic matroid* M.

2. Let V be a vector space over a field k, and A a finite set of vectors.

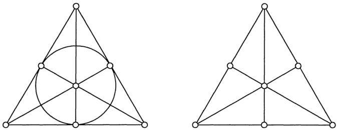
   Call a subset of A independent if it is linearly independent.

   This defines a matroid M *realizable* over k.

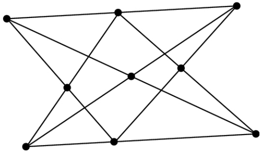




Fano matroid is realizable iff $\mathrm{char}(k) = 2$.



Non-Fano matroid is realizable iff $\mathrm{char}(k) \neq 2$.



Non-Pappus matroid is not realizable over any field.

How many matroids are realizable over a field?

0% of matroids are realizable.

In other words, almost all matroids are (conjecturally) not realizable over any field.

Testing the realizability of a matroid over a given field is not easy.

When $k = \mathbb{Q}$, this is equivalent to Hilbert's tenth problem over $\mathbb{Q}$ (*Sturmfels*).

When $k = \mathbb{R}, \mathbb{C},$ etc, there are universality theorems on realization spaces.

One can define the *characteristic polynomial* of a matroid by the recursion

$$\chi_M(q) = \chi_{M \setminus e}(q) - \chi_{M/e}(q).$$

Rota-Welsh conjecture (1970)

The coefficients of the characteristic polynomial $\chi_M(q)$ form a log-concave sequence for any matroid M, that is,

$$\mu_i^2 \geq \mu_{i-1}\mu_{i+1} \quad \textit{for all } i.$$

This implies the conjecture on G and the conjecture on A (*Brylawski*).

When M is realizable over a field k, the conjecture can be proved using results from algebraic geometry over k, but not in the general case.

A matroid M on E can be viewed as an r-dimensional fan in an n-dimensional space

$$\Sigma_M \subseteq \mathbb{R}^E/\mathbb{R}$$

whose maximal cones correspond to flags of flats $\varnothing \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r \subsetneq E$.

More precisely, the maximal cones are of the form

$$\text{cone}(\mathbf{e}_{F_1}, \dots, \mathbf{e}_{F_r}), \qquad \mathbf{e}_F = \sum_{i \in F} \mathbf{e}_i \in \mathbb{R}^E/\mathbb{R}.$$

The simplicial fan Σ_M is the *tropical linear space* associated to M.

In a recent joint work with Karim Adiprasito and Eric Katz, we obtained inequalities

that imply Rota-Welsh conjecture in its full generality.

What we show is that the tropical variety Σ_M has a "cohomology ring"

which *has the structure of the cohomology ring of a smooth projective variety.*

There is a young Italian, Bombieri, who is working on zeta functions. He noticed all by himself that it was necessary to prove in all characteristics that the intersection form on "primitive" algebraic cycles of half dimension is definite; furthermore, he also apparently spotted the conjecture according to which the factors of an algebraic cycle in a "Künneth" decomposition are algebraic. By the way, what are you up to in these directions?

From Jean-Pierre Serre to Alexander Grothendieck, 1964.

Our argument is a good advertisement for tropical geometry to pure combinatorialists:

For any two matroids on E with the same rank, there is a diagram

$$\Sigma_M \xrightarrow{\text{“}\mathit{flip}\text{”}} \Sigma_1 \xrightarrow{\text{“}\mathit{flip}\text{”}} \Sigma_2 \xrightarrow{\text{“}\mathit{flip}\text{”}} \cdots \xrightarrow{\text{“}\mathit{flip}\text{”}} \Sigma_{M'} \ ,$$

where “*flip*” is a tropical version of the classical flip, a local modification of Σ

that preserves the validity of the “*standard conjectures*” in their cohomology rings.

The intermediate objects are tropical varieties with good cohomology rings,

but not in general associated to a matroid (unlike in McMullen’s case of polytopes).

The cohomology ring $A^*(\Sigma_M)$ can be described explicitly by generators and relations, which can be taken as a definition.

Definition

The cohomology ring of Σ_M is the quotient of the polynomial ring

$$A^*(\Sigma_M) := \mathbb{Z}[x_F]/(I_1 + I_2),$$

where the variables are indexed by nonempty proper flats of M, and

$$\begin{aligned} I_1 &:= \text{ideal}\left(x_{F_1} x_{F_2} \mid F_1 \text{ and } F_2 \text{ are incomparable flats of } M \right), \\ I_2 &:= \text{ideal}\left(\sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F \mid i_1 \text{ and } i_2 \text{ are distinct elements of } E \right). \end{aligned}$$

Theorem

The cohomology ring of Σ_M is a Poincaré duality algebra of dimension r:

(1) *Degree map: There is an isomorphism*

$$\mathit{deg} : A^r(\Sigma_M) \simeq \mathbb{Z}, \qquad \prod_{i=1}^{r} x_{F_i} \longmapsto 1,$$

for any complete flag of flats $\varnothing \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r \subsetneq E$.

(2) *Poincaré duality: For any nonnegative integer $k \leq r$, the multiplication defines an isomorphism*

$$A^k(\Sigma_M) \simeq \mathit{Hom}_{\mathbb{Z}}(A^{r-k}(\Sigma_M), A^r(\Sigma_M)).$$

Note that the underlying simplicial complex of Σ_M, the *order complex* of M, is almost never Gorenstein.

Definition 3.1. Let ℓ be a piecewise linear function on Σ, and let σ be a cone in Σ.

(1) The function ℓ is *convex* around σ if it is equivalent to a piecewise linear function that is zero on σ and nonnegative on the link of σ.

The classes of *strictly* convex piecewise linear functions on Σ define the *ample cone*

$\mathscr{K}_\Sigma \subseteq A^1(\Sigma)_\mathbb{R} :=$ {piecewise linear functions on Σ}/{linear functions on Σ}.

Main Theorem

Let ℓ be an element of $\mathscr{K}_{\Sigma_M}$ and let k be a nonnegative integer $\leq r/2$.

(1) *Hard Lefschetz: The multiplication by ℓ defines an isormophism*

$$A^k(\Sigma_M)_{\mathbb{R}} \longrightarrow A^{r-k}(\Sigma_M)_{\mathbb{R}}, \qquad h \longmapsto \ell^{r-2k} \cdot h.$$

(2) *Hodge standard: The multiplication by ℓ defines a definite form of sign $(-1)^k$*

$$PA^k(\Sigma_M)_{\mathbb{R}} \times PA^k(\Sigma_M)_{\mathbb{R}} \longrightarrow A^r(\Sigma_M)_{\mathbb{R}} \simeq \mathbb{R}, \qquad (h_1, h_2) \longmapsto \ell^{r-2k} \cdot h_1 \cdot h_2,$$

where $PA^k(\Sigma_M)_{\mathbb{R}} \subseteq A^k(\Sigma_M)_{\mathbb{R}}$ is the kernel of the multiplication by ℓ^{r-2k+1}.

Why does this imply the log-concavity conjectures? Essentially because

$$b^2 \geq ac \text{ if and only if } \begin{vmatrix} b & a \\ c & b \end{vmatrix} \geq 0.$$