Plethysm and lattice point counting

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joint work with M. Michałek

Representation theory

How can a group act linearly on a (finite-diml.) vector space?

- Study homomorphisms $G \to GL(V)$
- Here: $G = GL_n$, so we look at $GL_n \to GL_N$.

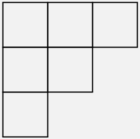
Examples

- Trivial representation $GL_n \to GL_1, g \mapsto 1$
- Determinant representation $GL_n \to GL_1$, $g \mapsto \det(g)$
- Standard Representation $GL_n \to GL_n$, $g \mapsto g$.
- Tensor product, symmetric powers, wedge powers, etc.
- Weird stuff (like $\mathbb{C}^*$ automorphisms) (we'll avoid these).

Irreducible representations

- A representation is reducible if there is some subspace $W \subseteq V$ that is left invariant by G (a subrepresentation).
- The best groups are reductive: any representation is a direct sum of irreducible representations.
- GL_n is a reductive group.
- First order of business: Understand irreducibles.

- Finite-dimensional, rational, irreducible representations of GL_n are indexed by Young diagrams with at most n rows, or equivalently, partitions of integers with at most n parts.



$6 = 3 + 2 + 1$

Example and Convention

- Only one row ($\lambda = d$): $S^\lambda W = S^d W$ (degree d forms on W^*)
- Only one column ($\lambda = 1 + \cdots + 1$)): $S^\lambda(W) = \bigwedge^d W$

- Taking symmetric powers, or $\bigwedge$ is a functor!
- In fact, for any λ there is a Schur functor S^λ.
- S^λ applied to the standard representation gives an irreducible representation (also denoted S^λ).

After understanding irreducibles...

... how do reducible representations decompose into irreducibles?

- Littlewood-Richardson: Decompose $S^\mu \otimes S^\nu$
- Plethysm: Decompose $S^\mu(S^\nu)$

The general plethysm

- $S^{\mu}(S^{\nu})$ decomposes into $\bigoplus_{\lambda}(S^{\lambda})^{\oplus c_{\lambda}}$.
- General plethysm: Determine c_{λ} as a function of (λ, μ, ν).

$\rightarrow$ impossible?

More modest goals – Symmetric powers

- Decompose $S^{d}(S^{k}W)$ into irreducibles ($d, k \in \mathbb{N}$). $\rightarrow$ still hard.
- Low degree: Fix d, seek function of (k, λ).

Proposition (Thrall, 1942)

One has $GL(W)$-module decompositions

$$S^2(S^kW) = \bigoplus S^\lambda W, \qquad \bigwedge^2(S^kW) = \bigoplus S^\mu W, \qquad \text{where}$$

- λ runs over tableaux with $2k$ boxes in two rows of even length.
- μ runs over tableaux with $2k$ boxes in two rows of odd length.

- Note: Divisibility conditions on the appearing tableaux.
- Similar formulas for $S^3(S^k)$ obtained by Agaoka, Chen, Duncan, Foulkes, Garsia, Howe, Plunkett, Remmel, Thrall, ...
- A few things are known about $S^4(S^k)$ (Duncan, Foulkes, Howe).
- Observation: Tableaux counting formulas get unwieldy quickly.

Theorem (KM14)

Fix d. For any $k \in \mathbb{N}$, and $\lambda \vdash dk$, the multiplicity of S^λ in $S^d(S^k)$ is

$$(-1)^{\binom{d-1}{2}} \left(\sum_{\alpha \vdash d} \frac{D_\alpha}{d!} \sum_{\pi \in S_{d-1}} \operatorname{sgn}(\pi) Q_\alpha(k, \lambda_\pi) \right),$$

where

- Q_α are counting functions of parametric lattice polytopes
- D_α is the number of permutations of cycle type α.
- $\lambda_\pi = (\lambda_1 + d - 1 - \pi(1), \lambda_2 + d - 2 - \pi(2), \ldots, \lambda_{d-2} + 2 - \pi(d-2))$

$\rightarrow$ Arithmetics with shifted lattice point enumerators.

Lattice point counting

Let $P \subseteq \mathbb{R}^d$ be a rational polytope. The Ehrhart function is

$$\phi : k \mapsto \#(kP \cap \mathbb{Z}^d)$$

- The Ehrhart function is a quasipolynomial.
  - Polynomial with periodic functions as coefficients
  - Polynomial in floor functions of linear terms

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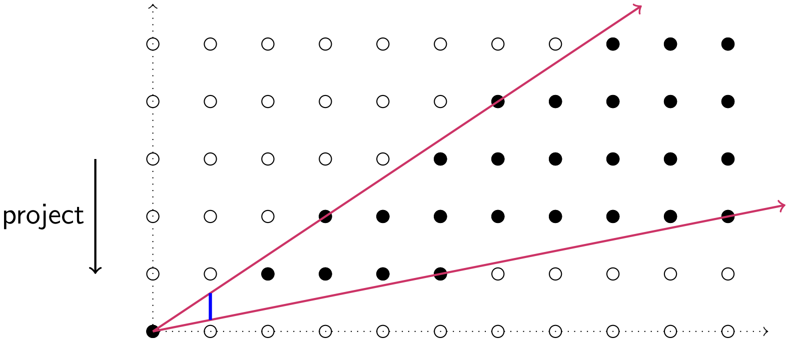
- Degree equals dimension of P.
- Ehrhart got his PhD when he was 60.

Lattice point counting



$$\phi(k) = 1 \quad 0 \quad 1 \quad 2 \quad 2 \quad 3 \quad 3 \quad 3 \quad 4 \quad 5 \quad 5$$

$$P = \left[\frac{1}{5}, \frac{2}{3}\right], \qquad \phi(k) = \begin{cases} 0 & k < 0, \\ (k+1) - \lfloor \frac{k+2}{3} \rfloor - \lfloor \frac{k+4}{5} \rfloor & k \geq 0. \end{cases}$$

Parametric lattice point counting is the multivariate generalization.

General setup

- Consider any (linear) projection of pointed rational cones.
- For each lattice point in the image, count fiber polytope
- One Ehrhart qpoly along each ray in the image cone.

Parametric lattice point counting

A piecewise quasipolynomial is a

- decomposition of space into polyhedral chambers
- quasipolynomial in each chamber
- continuity on the boundaries

Some facts

- Only finitely many combinatorial types of fibers (chambers).
- Blakley/Sturmfels: Lattice point enumerator is a pw. qpoly.
- Brion-Vergne connect lattice point enumerator to volume.
- Algorithms well-developed due to CS applications.

The counting functions Q_α

Definition: (α, λ, k)-matrices

Let α, λ be partitions with a and $d-1$ parts, respectively.
An (α, λ, k)-matrix is a matrix $M \in \mathbb{N}^{a \times (d-1)}$ with

- each row sums to k,
- the α-weighted entries of the j-th column sum to λ_j.

$\Rightarrow$ Let $Q_\alpha(k, \lambda)$ be the number of (α, λ, k)-matrices.

- This is a piecewise quasipolynomial in $k, \lambda_1, \ldots, \lambda_d$.

$\alpha = (1, \ldots, 1)$ yields integer points in a transportation polytope.

Theorem (KM14)

For $k \in \mathbb{N}$, and λ partition of dk, the multiplicity of S^λ in $S^d(S^k)$ is

$$(-1)^{\binom{d-1}{2}} \left(\sum_{\alpha \vdash d} \frac{D_\alpha}{d!} \sum_{\pi \in S_{d-1}} \operatorname{sgn}(\pi) Q_\alpha(k, \lambda_\pi) \right),$$

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Idea of the proof

Counting monomials in the character of the plethysm.

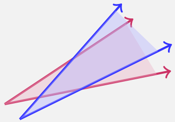
Evaluation with computer algebra

We used BARVINOK/ISL (Verdoolaege) + scripts to do $d = 3, 4, 5$.

`http://www.thomas-kahle.de/plethysm.html`

E.g. $d = 5$: 41 MB text or 3.6 MB entropy compressed.

- Actively developed software with very helpful mailing-list
- BUT: This is not a CAS for humans.
  - Almost no simplification routines implemented.
  - Takes 6 hours to read a qpoly that takes seconds to write.
  - BARVINOK returns parametric sets of constant functions ...
- There are too many chambers (partially our fault)
  - Extra chambers even for $d = 3$



Fewer errors

Quote from [Howe, 87] *Here we will outline what is involved in the computations and list our answers. The details are available from the author on request. The author does hope someone will check the calculations, because he does not have a great deal of faith in his ability to carry through the details in a fault-free manner. He hopes however that the answers are qualitatively correct as stated.*

Evaluation is quick

The multiplicity of the isotypic component of

$\lambda = (616036908677580244, 1234567812345678, 12345671234567, 123456123456)$

in $S^5(S^{123456789123456789})$ equals

$$240963570406235277976739158010615905293817243845463524159304407436599680700016051.$$

$\Rightarrow$ Much faster than finding values in Russian nuclear physics tables from the 70s.

Parametric evaluation is quick

Let $\lambda = (31, 3, 2, 2, 2)$. The multiplicity of $S^{s\lambda}$ in $S^5(S^{8s})$ equals

$$A(s) + \begin{cases} 1 & \text{if } s \equiv 0 \mod 5 \\ \frac{3}{5} & \text{if } s \equiv 1 \mod 5 \\ \frac{4}{5} & \text{if } s \equiv 2, 3, 4 \mod 5, \end{cases}$$

where

$$p_1 = \frac{1}{720}s^3 + \frac{1}{20}s^2 - \frac{289}{720}s$$

$$p_2 = \frac{1}{8}s + \frac{5}{8}, \qquad p_3 = -\frac{1}{6}s + \frac{1}{3} \qquad p_4 = -\frac{1}{3}s + \frac{7}{12}$$

$$A(s) = p_1 + p_2\left\lfloor \frac{s}{2} \right\rfloor + p_3\left\lfloor \frac{s}{3} \right\rfloor + \left(p_4 + \frac{1}{2}\left\lfloor \frac{s}{3} \right\rfloor\right)\left\lfloor \frac{1+s}{3} \right\rfloor + \frac{1}{4}\left(\left\lfloor \frac{1+s}{3} \right\rfloor^2 + \left\lfloor \frac{s}{4} \right\rfloor - \left\lfloor \frac{3+s}{4} \right\rfloor\right)$$

Note: This is an honest quasipolynomial!
(i.e. not piecewise)

The magic of singular reduction and quantization

Meinrenken-Sjamaar theory (advertised by M. Vergne)

- $[Q, R] = 0$
- Quasipolynomials are conical $\Rightarrow$ No small chambers.
- Assumptions?



The magic of cancellation

$$(-1)^{\binom{d-1}{2}}\left(\sum_{\alpha\vdash d}\frac{D_\alpha}{d!}\sum_{\pi\in S_{d-1}}\operatorname{sgn}(\pi)Q_\alpha(k,\lambda_\pi)\right)$$

Term for $\alpha=(1,\ldots,1)$ is a lattice point enumerator.

$$=\frac{\chi_\mu(1,\ldots,1)}{d!}\#P^\lambda_{k,d}+(-1)^{\binom{d-1}{2}}\left(\sum_{\alpha\vdash d,\alpha\neq(1,\ldots,1)}\ldots\right),$$

for an explicit polytope $P^\lambda_{k,d}$ (different from Q_α counted polytopes!).

Asymptotics

Explicit formulas allow to study asymptotics.

- Roger Howe identified leading terms of $S^3(S^k)$ and $S^4(S^k)$
- Howe's conjecture: Leading term comes from $P^{\lambda}_{k,d}$.

Leading terms of plethysm

- The lattice polytopes counted by Q_α have different dimensions.
- Highest contribution from $\alpha = (1, 1, \ldots, 1)$?
- Not obvious because of cancellation!

Theorem (KM14)

The multiplicity of the isotypic component corresponding to λ inside $S^d(S^k(V))$ is a piecewise quasipolynomial in k and λ. In each full-dimensional conical chamber its highest degree term equals $\frac{\dim \mu}{d!}$ times the highest degree term of the multiplicity of λ in $S^k(V)^{\otimes d}$.

Open problem

- What is the leading term along rays with $\lambda_i = \lambda_{i+1}$?

Combinatorial formulas

Quote from [Stanley, 99]: *Often coefficients [in expansions of symmetric polynomials] have representation-theoretic interpretation as a multiplicity, providing a proof of non-negativity. If this is the only known proof of non-negativity, then the problem is to find a combinatorial proof.*

Prime example: The Littlewood-Richardson rule for $S^\lambda \otimes S^\mu$

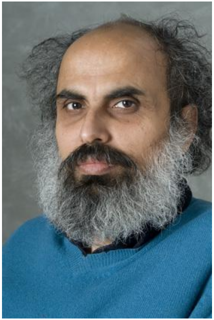
Problem 9

Find a combinatorial interpretation of plethysm coefficients, thereby combinatorially reproving that they are non-negative.

Mulmuley's question

Fix λ. Is the multiplicity of $S^{s\lambda}$ in $S^d(S^{sk})$ an Ehrhart function of a rational polytope?

The answer is no, even for regular λ.



Example (KM15)

After reductions: Multiplicity function in $S^3(S^k)$ has two parameters: (k,b) ($b=\lambda_2$, λ_1 determined, $\lambda_3=0$ can be assumed).

- Domain is 2d bounded by $b \geq 0$ and $3k \geq 2b$.
- Chamber split along $k=b$.
- Periodicity of quasipolynomial is 6, growth is linear.
- On boundary rays like $(k,b)=(k,0)$: $1,0,1,0,1,0,1,0,\dots$

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- $(k, b) = (14 \cdot l, 17 \cdot l)$:

$$\phi(l) = 1, 1, 3, 4, 6, 6, 9, 9, \ldots$$

Is this an Ehrhart function of a rational interval?

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  Is this an Ehrhart function of a rational interval?
- No!: Ehrhart Reciprocity violated: $\phi(-1) = -2$
- Computers extremely helpful in this kind of investigation.

Refinements

We can actually do all of this for arbitrary outer partition $\mu \vdash d$:
The multiplicity of S^{λ} in $S^{\mu}(S^{k})$ equals

$$(-1)^{\binom{d-1}{2}}\left(\sum_{\alpha\vdash d}\chi_{\mu}(\alpha)\frac{D_{\alpha}}{d!}\sum_{\pi\in S_{d-1}}\operatorname{sgn}(\pi)Q_{\alpha}(k,\lambda_{\pi})\right),$$

where χ_{μ} is a character permutation group S_d.

Outlook: lattice point counting in representation theory

Foulkes conjecture

If $a < b$ then $S^a(S^b)$ embeds as a subrepresentation into $S^b(S^a)$.

- Known up to $a = 4$.
- Need formulas for fixed inner partition.
- Results of Bedratyuk indicate this may be possible.
- Need to decide positivity of a difference of two pw. qpolys.

- Kronecker coefficients (Tensor products of Specht modules)?

Tweaking the computation

- Compute chamber decomposition of result a priori.

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