Binomial edge ideals and determinantal facet ideals

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Binomial edge ideals

Let G be a finite simple graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$. Associated to G is a binomial ideal

$$J_G = (f_{ij} : i < j, \ \ \{v_i, v_j\} \in E(G)),$$

in $S = k[x_1, \dots, x_n, y_1, \dots, y_n]$, called the binomial edge ideal of G, in which $f_{ij} = x_i y_j - x_j y_i$.

It could be seen as the ideal generated by a collection of 2-minors of a $(2 \times n)$-matrix whose entries are all indeterminates.

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Reduced Gröbner basis

By $<$, we mean the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$.

Herzog - Hibi - Hreinsdóttir - Kahle - Rauh (2010)

Let G be a graph. Then $\mathrm{in}_< J_G$ is a squarefree monomial ideal. In particular, J_G is a radical ideal.

Minimal primes

Let G be a graph $[n]$, and let $G_1, \ldots, G_{c(T)}$ be the connected component of $G_{[n]\setminus T}$, the induced subgraph of G on $[n] \setminus T$. For each G_i we denote by $\widetilde{G}_i$ the complete graph on the vertex set $V(G_i)$. For each subset $T \subset [n]$ a prime ideal $P_T(G)$ is defined as

$$P_T(G) = (\bigcup_{i \in T} \{x_i, y_i\}, J_{\widetilde{G}_1}, \ldots, J_{\widetilde{G}_{c(T)}}).$$

Minimal primes

Herzog - Hibi - Hreinsdóttir - Kahle - Rauh (2010)

Let G be a graph $[n]$. Then $J_G = \bigcap_{T\subset[n]} P_T(G)$.

Herzog - Hibi - Hreinsdóttir - Kahle - Rauh (2010)

Let G be a graph $[n]$. Then $P_T(G)$ is a minimal prime ideal of J_G if and only if $T = \emptyset$, or each $i \in T$ is a cut point of the graph $G_{([n]\setminus T)\cup\{i\}}$.

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Corollary

J_G is a prime ideal if and only if all connected components of G are complete graphs.

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Dimension

Corollary

Let G be a graph $[n]$. Then $\mathrm{height} P_T(G) = |T| + (n - c(T))$ and

$$\dim S/J_G = \max\{(n - |T|) + c(T) : T \subset [n]\}.$$

In particular, $\dim S/J_G \geq n + c$, where c is the number of connected components of G.

Closed graphs

Herzog - Hibi - Hreinsdóttir - Kahle - Rauh (2010)

The following conditions are equivalent:

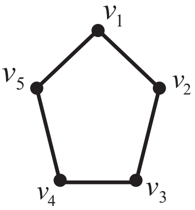
(1) The generators f_{ij} of J_G form a quadratic Gröbner basis.

(2) For all edges $\{i, j\}$ and $\{k, l\}$ with $i < j$ and $k < l$ one has $\{j, l\} \in E(G)$ if $i = k$, and $\{i, k\} \in E(G)$ if $j = l$.

Closed graphs

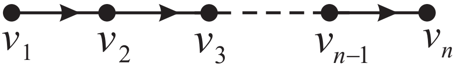
A graph G is said to be closed with respect to the given labeling of the vertices, if G satisfies conditions of previous theorem, and a graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$ is said to be closed, if its vertices can be labeled by the integer $1, 2, \ldots, n$ such that for this labeling G is closed.

Closed graphs



C_5 is not a closed graph.

Closed graphs



P_n is a closed graph.

Closed graphs

Ene - Herzog - Hibi (2010)

The following conditions are equivalent:

(1) G is closed.

(2) There exists a labeling of G such that all facets of the clique complex of G are intervals.

Graded Betti numbers

Ene - Herzog - Hibi (2010)

Let G be a closed graph with Cohen-Macaulay binomial edge ideal. Then $\beta_{ij}(J_G) = \beta_{ij}(\mathrm{in}_<(J_G))$ for all i, j.

Conjecture (Ene - Herzog - Hibi (2010))

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Linear resolutions

Suppose I is a homogeneous ideal of R whose generators all have degree d. Then I has a linear resolution if for all $i \geq 0$, $\beta_{i,j}(I) = 0$ for all $j \neq i + d$.

Linear resolutions

Kiani - SM (2012)

Let G be a graph with no isolated vertices. Then the following conditions are equivalent:

(1) J_G has a linear resolution.

(2) J_G is linearly presented.

(3) $\mathrm{in}_<(J_G)$ has a linear resolution.

(4) G is a complete graph.

Pure resolutions

Let I be a homogeneous ideal of S whose generators all have degree d. Then I has a d-pure resolution (or pure resolution) if its minimal graded free resolution is of the form

$$0 \to S(-d_p)^{\beta_p(I)} \to \cdots \to S(-d_1)^{\beta_1(I)} \to I \to 0,$$

where $d = d_1$.

Pure resolutions

Schenzel - Zafar (2014)

If G is a complete bipartite graph, then J_G has a pure resolution.

Pure resolutions

Kiani - SM (2014)

Let G be a graph with no isolated vertices. Then J_G has a pure resolution if and only if G is a :

(1) complete graph, or

(2) complete bipartite graph, or

(3) disjoint union of some paths.

Regularity

Matsuda - Murai (2013)

Let G be a graph on $[n]$, and let ℓ be the length of the longest induced path in G. Then

$$\mathrm{reg}(J_G) \geq \ell + 1.$$

Denoted $c(G)$ we mean the number of maximal cliques of G.

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Ene - Zarojanu (2014)

Let G be a block graph. Then $\mathrm{reg}(J_G) \leq c(G) + 1$.

Ene - Zarojanu (2014)

Let G be a closed graph with connected components $G_1, \dots, G_r$. Then

$$\operatorname{reg}(J_G) = \operatorname{reg}(\operatorname{in}_<(J_G)) = \ell_1 + \cdots + \ell_r + 1,$$

where ℓ_i is the length of the longest induced path of G_i.

Regularity

Kiani - SM (2015)

Let G_1 and G_2 be graphs on $[n_1]$ and $[n_2]$, respectively, not both complete. Then

$$\operatorname{reg}(J_{G_1 * G_2}) = \max\{\operatorname{reg}(J_{G_1}), \operatorname{reg}(J_{G_2}), 3\}.$$

Corollary

Let G be a complete t-partite graph which is not complete. Then $\operatorname{reg}(J_G) = 3$.

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Let G be a graph on n vertices. Then $\operatorname{reg}(J_G) \leq n$.

Conjecture (Matsuda - Murai (2013))

Let $G \neq P_n$ be a graph on n vertices. Then $\operatorname{reg}(J_G) \leq n-1$.

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Let C_n be an n-cycle. Then $\operatorname{reg}(J_{C_n}) = n - 1$.

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Mohammadi - Sharifan (2014)

Let G be a graph and $e = \{i, j\}$ be an edge of G. Then

$$J_{G\setminus e} : f_e = J_{(G\setminus e)_e} + I_G,$$

where

$$I_G = (g_{P,t} \; : \; P : i, i_1, \ldots, i_s, j \text{ and } 0 \leq t \leq s),$$

$g_{P,0} = x_{i_1} \cdots x_{i_s}$ and $g_{P,t} = y_{i_1} \cdots y_{i_t} x_{i_{t+1}} \cdots x_{i_s}$ for every $1 \leq t \leq s$.

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring. We view S as a standard graded K-algebra by assigning to each x_i the degree 1. A graded complex

$$\mathbb{G}: \quad \cdots \to G_2 \to G_1 \to G_0 \to 0$$

of finitely generated graded free S-modules is called a linear complex (with initial degree d) if for all i, $G_i = S(-i-d)^{b_i}$ for suitable integers b_i.

The linear strand

Let M be a finitely generated graded S-module, and let d be the initial degree of M, and let $(\mathbb{F}, \partial)$ be the minimal graded free resolution of M with $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$. Note that $\beta_{ij} = 0$ for all pairs (i, j) with $j < i + d$.

Let F_i^{lin} be the direct summand $S(-i-d)^{\beta_{i,i+d}}$ of F_i. It is obvious that $\partial(F_i^{\mathrm{lin}}) \subset F_{i-1}^{\mathrm{lin}}$ for all $i > 0$.

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Thus

$$\mathbb{F}^{\text{lin}} : \cdots \longrightarrow F_2^{\text{lin}} \longrightarrow F_1^{\text{lin}} \longrightarrow F_0^{\text{lin}} \longrightarrow 0$$

is a subcomplex of $\mathbb{F}$, called the linear strand of the resolution of M.

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The linear strand

Denoted by $(f_0(\Delta), f_1(\Delta), \ldots, f_d(\Delta))$ is the f-vector of a d-dimensional simplicial complex Δ.

Conjecture (Kiani - SM (2014))

Let G be a graph. Then $\beta_{i,i+2}(J_G) = (i+1)f_{i+1}(\Delta(G))$, where $\Delta(G)$ is the clique complex of G.

Determinantal facet ideal

A clutter C on the vertex set $[n]$ is a collection of subsets of $[n]$ with no containment between its elements. An element of C is called a circuit. If all circuits of C have the same cardinality m, then C is called an m-uniform clutter.

A clique of an m-uniform clutter C is a subset σ of $[n]$ such that each m-subset of σ is a circuit of C. We denote by $\Delta(C)$ the simplicial complex whose faces are the cliques of C which is called the clique complex of C.

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Determinantal facet ideal

Let C be an m-uniform clutter on $[n]$. To each circuit $\tau \in C$ with $\tau = \{j_1, \ldots, j_m\}$ and $1 \leq j_1 < j_2 < \cdots < j_m \leq n$ we assign the m-minor $\mathbf{m}_\tau$ of $X = (x_{ij})$ which is determined by the columns $1 \leq j_1 < j_2 < \cdots < j_m \leq n$.

Denoted by J_C is the ideal in $S = K[x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$ which is generated by the minors $\mathbf{m}_\tau$ with $\tau \in C$. This ideal is called the determinantal facet ideal of C.

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The linear strand

Herzog - Kiani - SM (2015)

Let $\mathbb{G}$ be a finite linear complex with initial degree d. Then the following conditions are equivalent:

(1) $\mathbb{G}$ is the linear strand of a finitely generated graded S-module with initial degree d.

(2) $H_i(\mathbb{G})_{i+d+j} = 0$ for all $i > 0$ and for $j = 0, 1$.

Eagon-Northcott complex

Let F and G be free S-modules of rank m and n, respectively, with $m \leq n$, and let $\varphi : G \to F$ be an S-module homomorphism.

We choose a basis $f_1, \ldots, f_m$ of F and a basis $g_1, \ldots, g_n$ of G. Let $\varphi(g_j) = \sum_{i=1}^{m} \alpha_{ij} f_i$ for $j = 1, \ldots, n$. The matrix $\alpha = (\alpha_{ij})$ describing φ with respect to these bases is an $(m \times n)$-matrix with entries in S.

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The ideal of m-minors of this matrix is denoted $I_m(\varphi)$. It is know that if $\mathrm{grade}\, I_m(\varphi) = n - m + 1$, then the so-called Eagon-Northcott complex provides a free resolution of $I_m(\varphi)$.

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Denote by $S(F)$ is the symmetric algebra of F. The complex

$$\mathcal{C}(\varphi):\ 0 \to \bigwedge^{n} G \otimes S_{n-m}(F)^* \to \cdots \to \bigwedge^{m} G \otimes S_0(F)^* \to 0,$$

is called the Eagon-Northcott complex.

Eagon-Northcott complex

We set $\mathcal{C}_i(\varphi) = \bigwedge^{m+i} G \otimes S_i(F)^*$ and $\mathbf{b}(\sigma; \mathbf{a}) = g_\sigma \otimes f^{(\mathbf{a})}$, where $g_\sigma = g_{j_1} \wedge \cdots \wedge g_{j_{m+i}}$ for $\sigma = \{j_1 < j_2 < \cdots < j_{m+i}\}$, and $f^{(\mathbf{a})}$ is the dual of $f^{\mathbf{a}} = f_1^{a_1} f_2^{a_2} \cdots f_m^{a_m}$ with $a \in \mathbb{Z}^m_{\geq 0}$ and $|a| = a_1 + \cdots + a_m = i$. Moreover, we set $f^{(\mathbf{a})} = 0$ if $a_i < 0$ for some i.

Then the elements $\mathbf{b}(\sigma; \mathbf{a})$ form a basis of $\mathcal{C}_i(\varphi)$, and

$$\partial(\mathbf{b}(\sigma; \mathbf{a})) = \sum_{k=1}^{m+i} \sum_{\ell=1}^{m} (-1)^{k+1} \alpha_{\ell j_k} \mathbf{b}(\sigma \setminus \{j_k\}; \mathbf{a} - \mathbf{e}_\ell).$$

Here $\mathbf{e}_1, \ldots, \mathbf{e}_m$ is the canonical basis of $\mathbb{Z}^m$.

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Generalized Eagon-Northcott complex

Let Δ be a simplicial complex on $[n]$. We denote $\mathcal{C}_i(\Delta;\varphi)$ the free submodule of $\mathcal{C}_i(\varphi)$ generated by all $\mathbf{b}(\sigma;\mathbf{a})$ such that $\sigma \in \Delta$ with $|\sigma| = m+i$, and $\mathbf{a} \in \mathbb{Z}^m_{\geq 0}$ with $|\mathbf{a}| = i$.

Since $\partial(\mathbf{b}(\sigma;\mathbf{a})) \in \mathcal{C}_{i-1}(\Delta;\varphi)$ for all $\mathbf{b}(\sigma;\mathbf{a}) \in \mathcal{C}_i(\Delta;\varphi)$, we obtain the subcomplex

$$\mathcal{C}(\Delta;\varphi):\ 0 \to \mathcal{C}_{n-m}(\Delta;\varphi) \to \cdots \to \mathcal{C}_1(\Delta;\varphi) \to \mathcal{C}_0(\Delta;\varphi) \to 0$$

of $\mathcal{C}(\varphi)$ which we call the generalized Eagon-Northcott complex attached to the simplicial complex Δ and the module homomorphism $\varphi : G \to F$.

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of $\mathcal{C}(\varphi)$ which we call the generalized Eagon-Northcott complex attached to the simplicial complex Δ and the module homomorphism $\varphi : G \to F$.

Let X be an $(m \times n)$-matrix of indeterminates x_{ij}, and let S be the polynomial ring over a field K in the variables x_{ij}. Moreover, let $\varphi : G \to F$ be the S-module homomorphism of free S-modules given by the matrix X.

Now we give a $(\mathbb{Z}^m \times \mathbb{Z}^n)$-grading to the polynomial ring S, by setting $\mathrm{mdeg}(x_{ij}) = (e_i, \varepsilon_j)$ where e_i is the i-th canonical basis vector of $\mathbb{Z}^m$ and ε_j is the j-th canonical basis vector of $\mathbb{Z}^n$.

Generalized Eagon-Northcott complex as a linear strand

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The chain complex $\mathcal{C}(\Delta;\varphi)$ inherits this grading. More precisely, for each i, the degree of a basis element $\mathbf{b}(\sigma;\mathbf{a})$ of $\mathcal{C}_i(\Delta;\varphi)$ with $\sigma=\{j_1,\ldots,j_{m+i}\}$ is set to be $(\mathbf{a}+\mathbf{1},\gamma)\in\mathbb{Z}^m\times\mathbb{Z}^n$, where $\gamma=\varepsilon_{j_1}+\cdots+\varepsilon_{j_{m+i}}$, and $\mathbf{1}$ is the vector in $\mathbb{Z}^m$ whose entries are all equal to 1.

Generalized Eagon-Northcott complex as a linear strand

Herzog - Kiani - SM (2015)

Let Δ be a simplicial complex, and let m be a positive integer. Then the following conditions are equivalent:

(1) $\mathcal{C}(\Delta; \varphi)$ is the linear strand of a finitely generated graded S-module with initial degree m.

(2) Δ has no minimal nonfaces of cardinality $\geq m + 2$.

The linear strand of J_C

Herzog - Kiani - SM (2015)

Let C be an m-uniform clutter, and let $\mathbb{F}$ be the minimal graded free resolution of J_C. Then

$$\mathbb{F}^{\mathrm{lin}} \cong \mathcal{C}(\Delta(C); \varphi).$$

The linear strand of J_C

Corollary

Let C be an m-uniform clutter. Then

$$\beta_{i,i+m}(J_C) = \binom{m+i-1}{m-1} f_{m+i-1}(\Delta(C)),$$

for all i.

Therefore, the length of the linear strand of J_C is equal to

$$\dim \Delta(C) - m + 1,$$

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Herzog - Kiani - SM (2015)

Let C be an m-uniform clutter. Then the following conditions are equivalent:

(1) J_C has a linear resolution.

(2) J_C is linearly presented.

(3) C is a complete clutter.

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Thanks for your attention.