The poset of normal polytopes

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Combinatorial and Experimental Methods in Commutative Algebra and Related Fields

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Polytopes

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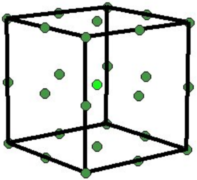
Fix a lattice $M \simeq \mathbb{Z}^n \subset \mathbb{R}^n$ A *lattice polytope* is a (convex hull of a) finite set of points in M.



(from Polymake page - great polytope software)

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One associates the graded algebra over the monoid to the polytope.

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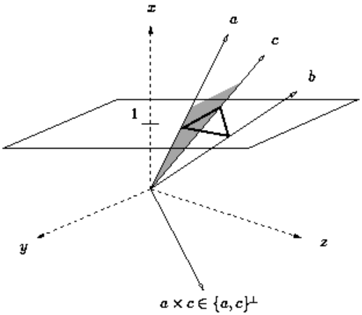

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A lattice polytope $P \subset M$ is normal if for every $k \in \mathbb{N}$ every lattice point in kP is a sum of k lattice points of P.

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Be careful with other definitions for different lattices!

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1. Correspond to (projectively normal embedded) toric varieties
2. Are good discrete counterparts of convex sets:

$$kP \cap M = (P \cap M) + \cdots + (P \cap M).$$

Algebra and combinatorics

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Normality $\leftrightarrow$ Projective Normality

General questions

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3. Can one make induction on normal polytopes?
4. Can one formally state which normal polytopes are more usual than the others?

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The very existence of minimal normal polytopes is nonobvious: a lot of normal polytopes can be reduced to points by consequently removing vertices (and not leaving the category of normal polytopes).
The story has a happy ending: minimal polytopes were found and indeed provided interesting counterexamples
[Normality and covering properties of affine semigroups. J. Reine Angew. Math.]

Poset structure

Definition ($NPol(d)$, Quantum jump)

We define the poset structure on the set of d dimensional normal polytopes in $\mathbb{Z}^d$ as the transitive closure of the relation $P < Q$ if $P \subset Q$ and Q contains only one lattice point not in P.
Such an elementary relation as above is called a quantum jump.

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Given the poset structure, a natural question arises:
Do there exist maximal normal polytopes?

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Natural questions:

1. Is $NPol(d)$ connected?
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3. What are the homology groups?

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Theorem

Let $P \subset \mathbb{R}^d$ be a lattice d-polytope and let $Q := conv(P, z)$ be an extension by one lattice point $z \in \mathbb{Z}^d$. If Q is normal then

$$|ht_F(z)| \leq 1 + (d-2)width_F P \tag{1}$$

for every facet F of P that is visible from z. This bound is sharp.

Proof.

Assume $z = 0$ and consider $(d-1)P$. Take a point in $\mathrm{Conv}(0, (d-1)P)$ just before $(d-1)F$.

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To prove the bound is sharp we present examples. □

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There are no maximal 3-dimensional simplices.

Open problems

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3. Is the convex hull of lattice points in any ball a normal polytope?

Theorem

The set of lattice points in any 3-dimensional ellipsoid forms a normal polytope.

Proof.

Let P be the convex hull of all lattice points in a 3-dimensional ellipsoid E centered at 0. Consider a lattice point $x \in 2P$.

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The tangent space to $2E$ at y defines a linear form l. Consider x_1 a vertex of P that maximizes the value of l.

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The difference $y - x_1$ seems to be near 0. Indeed, one can show that it belongs to E, hence also to P. We have $x = y_1 + (x - y_1)$, which is enough to conclude that P is normal. □

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1. Does there exist an even finer geometric structure on the space of normal polytopes? An analytic one?
2. If yes, can one encode the properties of quantum jumps like volume, as distance in that space?
3. Normal polytopes can be regarded as special cones. Does there exist a similar (smoother) space of cones, where the 'discrete order' coincides with the inclusion order?

Cones

In analogy to quantum jumps we may say that for two convex, rational polyhedral cones $C_1 < C_2$ if and only if

$$M \cap C_2 = (M \cap C_1) + \mathbb{Z}x$$

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Question

Is the transitive closure of $<$ simply the inclusion of cones?

We know the answer is positive in dimension three.

Thank you!