Computing symmetries of Mori dream spaces

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September 29, 2015

Outline

Outline:

1. Background on *Mori dream spaces*.
2. Computing symmetries
    - of graded algebras,
    - of Mori dream spaces.
3. Application.

Joint work with: Jürgen Hausen (Tübingen)

Background on Mori dream spaces

The *Cox ring* of a normal, projective variety X is the $\mathrm{Cl}(X)$-graded $\mathbb{C}$-algebra

$$\mathrm{Cox}(X) \;:=\; \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)).$$

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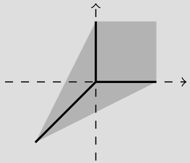
Example

For $X = \mathbb{P}_2$ we have $\mathrm{Cl}(\mathbb{P}_2) = \mathbb{Z}$ and

$$\mathrm{Cox}(\mathbb{P}_2) \ = \ \mathbb{C}[T_1, T_2, T_3], \qquad \deg(T_i) \ = \ 1 \ \in \ \mathbb{Z}.$$



Background on Mori dream spaces

We call X a *Mori dream space* if $\mathrm{Cl}(X)$ and $\mathrm{Cox}(X)$ are finitely generated.

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Example

The class of Mori dream spaces comprises

- toric varieties (Cox, Audin, Batyrev, Fine, Musson),
- rational complexity-one T-varieties (Knop, Hausen/Süß),
- smooth Fano varieties (Birkar/Cascini/Hacon/McKernan),
- ...

Mori dream spaces

Construction

Let X be a Mori dream space, $R := \mathrm{Cox}(X)$. There is a quotient

$$\mathrm{Spec}(R) \ = \ \overline{X} \ \supseteq \ \widehat{X} \xrightarrow{/\!\!/ H} X$$

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Remark: X completely determined by $(\mathrm{Cox}(X), \mathrm{Cl}(X), w)$.

Mori dream spaces

Example (Data fixing a Mori dream space)

1. **Cox ring and class group:** choose $\mathrm{Cl}(X) := \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ and

$$\mathrm{Cox}(X) := \mathbb{C}[T_1, \ldots, T_5]/\langle T_1 T_2 + T_3^2 + T_4^2 \rangle$$

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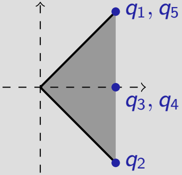

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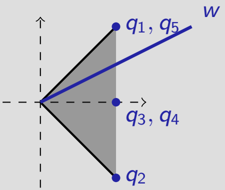

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❷ **Open set $\widehat{X}$:** fixed by choosing $w := (2, 1) \in \mathbb{Q}^2$.

Mori dream spaces: algorithms so far

Some algorithms for Mori dream spaces so far:

- Compute Cox rings of modifications. (HAUSEN/–/LAFACE, *Math. Comp., to appear*)
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X was the $A_3 2A_1$-singular Gorenstein log del Pezzo $\mathbb{C}^*$-surface, $\varrho(X) = 2$.

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Aim

Compute the group of *automorphisms* $\mathrm{Aut}(X)$.

Symmetries of Mori dream spaces

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$$R = \mathrm{Cox}(X), \quad \overline{X} = \mathrm{Spec}(R), \quad K = \mathrm{Cl}(X).$$

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Theorem (Arzhantsev/Hausen/Huggenberger/Liendo)

Exact sequence of linear algebraic groups:

$$\begin{array}{c} \mathrm{Aut}_K(R) \\ \wr\Vert \\ \mathrm{Aut}_H\left(\overline{X}\right) \\ \cup\vert \\ 1 \longrightarrow H \longrightarrow \mathrm{Aut}_H(\widehat{X}) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1 \end{array}$$

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$\rightsquigarrow$ *step 2* $\rightsquigarrow$ *step 3*

Step 1: $\mathrm{Aut}_K(R)$.

Graded algebras

Setting: Consider an affine, integral $\mathbb{C}$-algebra

$$R \;=\; \mathbb{C}[T_1, \ldots, T_r]/I \;=\; \bigoplus_{w \in K} R_w$$

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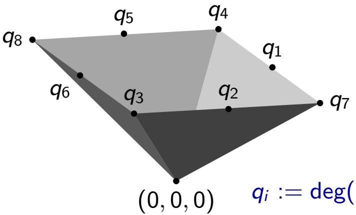

$q_i := \deg(T_i)$

Graded algebras

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The $\mathbb{C}$-algebra $R := \mathbb{C}[T_1, \ldots, T_5]/\langle T_1 T_2 + T_3^2 + T_4^2 \rangle$ is pointedly $K := \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$-graded via

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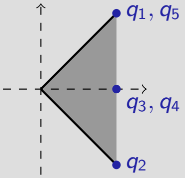

$\subseteq K \otimes \mathbb{Q} = \mathbb{Q}^2$

Graded algebras: symmetries

The *automorphism group* $\mathrm{Aut}_K(R)$ of a K-graded algebra R consists of all pairs (φ, ψ) with

- $\varphi\colon R \to R$ automorphism of $\mathbb{C}$-algebras,
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Aim

Represent $\mathrm{Aut}_K(R) \subseteq \mathrm{GL}(n, \mathbb{C})$ for some n.

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Proposition

❶ *Write* $G := \mathrm{Aut}_K(\mathbb{C}[T_1, \ldots, T_r])$*. There is an isomorphism*

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Algorithm ($\mathrm{Aut}_K(R)$)

1. Represent $\mathrm{Aut}_K(\mathbb{C}[T_1, \dots, T_r])$ as a subgroup $G \subseteq \mathrm{GL}(n, \mathbb{C})$.
2. For $(\varphi, \psi) \in G$, the condition $(\varphi, \psi) \cdot I_{q_i} = I_{\psi(q_i)}$ yields $J \subseteq \mathcal{O}(G)$ with

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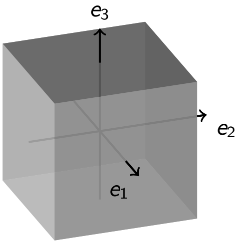
Task: compute lattice points



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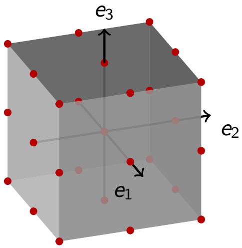

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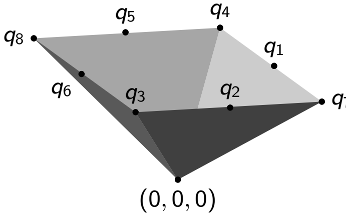

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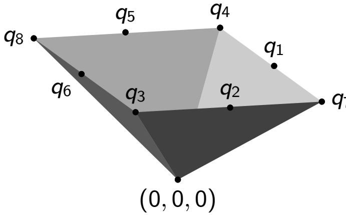

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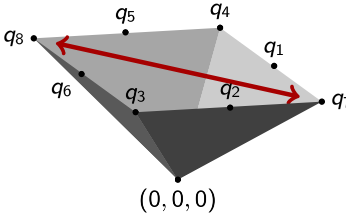

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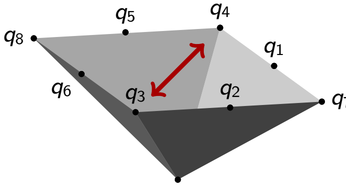

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Example (continued)

The group $\mathrm{Aut}_K(R)$ is isomorphic to the following subgroup of $\mathrm{GL}(5,\mathbb{C})$:

$$\left\{ \begin{bmatrix} Y_1 & 0 & 0 & 0 & 0 \\ 0 & Y_7 & 0 & 0 & 0 \\ 0 & 0 & Y_{13} & 0 & 0 \\ 0 & 0 & 0 & Y_{19} & 0 \\ 0 & 0 & 0 & 0 & Y_{25} \end{bmatrix} \in \mathrm{GL}(5,\mathbb{C});\quad \begin{array}{l} Y_{13}^2 = Y_{19}^2, \\ Y_1 Y_7 = Y_{13}^2 \end{array} \right\}$$

$$\cup \left\{ \begin{bmatrix} Y_1 & 0 & 0 & 0 & 0 \\ 0 & Y_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_{14} & 0 \\ 0 & 0 & Y_{18} & 0 & 0 \\ 0 & 0 & 0 & 0 & Y_{25} \end{bmatrix} \in \mathrm{GL}(5,\mathbb{C});\quad \begin{array}{l} Y_{14}^2 = Y_{18}^2, \\ Y_1 Y_7 = Y_{18}^2 \end{array} \right\}$$

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In particular:

$$\dim(\mathrm{Aut}_K(R)) = 3, \qquad \sharp \text{ components} = 4.$$

Step 2: $\mathrm{Aut}_H(\widehat{X})$.

Compute $\mathrm{Aut}_H(\widehat{X})$

Let X be a Mori dream space with

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1. As before, compute $G := \mathrm{Aut}_K(R) \subseteq \mathrm{GL}(n, \mathbb{C})$.
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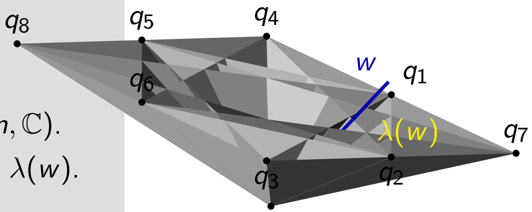

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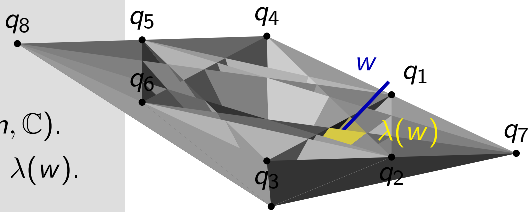

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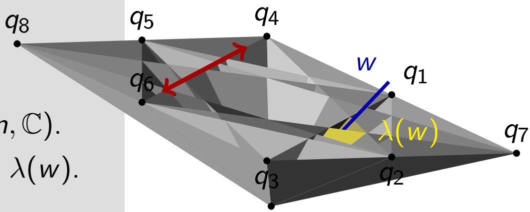


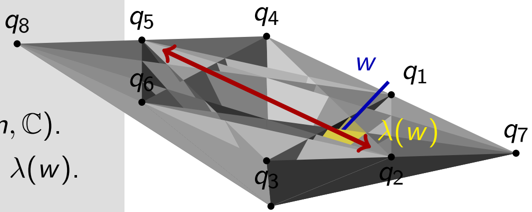
Compute $\mathrm{Aut}_H(\widehat{X})$

Let X be a Mori dream space with

$$R = \mathrm{Cox}(X), \quad K = \mathrm{Cl}(X), \quad w \in K \otimes \mathbb{Q} \text{ ample.}$$

Algorithm ($\mathrm{Aut}_H(\widehat{X})$)

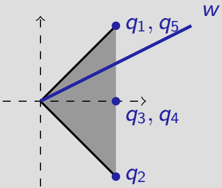
1. As before, compute $G := \mathrm{Aut}_K(R) \subseteq \mathrm{GL}(n, \mathbb{C})$.
2. Compute the *GIT-cone* $\lambda(w)$.
3. Return $\{(\varphi, \psi) \in G;\ \psi(\lambda(w)) = \lambda(w)\}$.



Compute $\mathrm{Aut}_H(\widehat{X})$

Example (continued)

For the previous example, w and the $q_i := \deg(T_i) \in K \otimes \mathbb{Q}$ were



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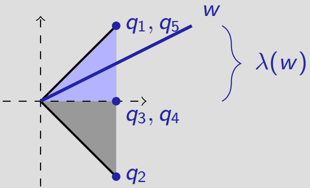

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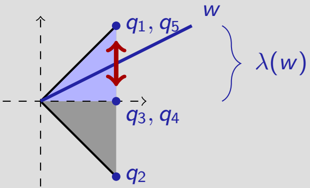

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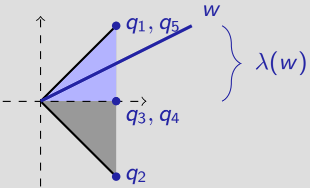

In particular: $\mathrm{Aut}_H(\widehat{X}) \cong \mathrm{Aut}_K(R)$.

Step 3: $\mathrm{Aut}(X)$

Compute $\mathrm{Aut}(X)$

Recall: X a Mori dream space, $H = \mathrm{Spec}\,\mathbb{C}[\mathrm{Cl}(X)]$. Then

$$\mathrm{Aut}(X) \;\cong\; \mathrm{Aut}_H(\widehat{X})/\,H.$$

Algorithm (coordinate ring of $\mathrm{Aut}(X)$)

1. Compute $G := \mathrm{Aut}_H(\widehat{X}) \subseteq \mathrm{GL}(n, \mathbb{C})$ and its $\mathrm{Cl}(X)$-graded coordinate ring $\mathbb{C}[Y_{ij}, Z]/J$.

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2. Determine generators $f_1, \ldots, f_m$ for $\mathbb{C}[Y_{ij}, Z]_0$ and generators for $I_1 := \Psi^{-1}(J)$ with

$$\Psi\colon \mathbb{C}[S_1, \ldots, S_m] \to \mathbb{C}[Y_{ij}, Z], \qquad S_i \mapsto f_i.$$

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$$\Psi\colon \mathbb{C}[S_1, \ldots, S_m] \to \mathbb{C}[Y_{ij}, Z], \qquad S_i \mapsto f_i.$$

3. $\mathcal{O}(\mathrm{Aut}(X)) \cong \mathbb{C}[S_1, \ldots, S_m]/(I_1 + \text{relations among } f_i)$.

Compute $\mathrm{Aut}(X)$

Example (continued)

For the Mori dream space X from before:

$$
\begin{aligned}
\mathrm{Aut}(X) \;&\cong\; V\left(\begin{array}{l}
T_{11},\, T_{10},\, T_7,\, T_5,\, T_9T_{15},\\
T_2 - T_9,\, T_1 - T_8,\, T_4 - T_6 + T_{12},\, T_{14} + T_{15} - 1,\\
T_3 - T_8 - T_9,\, T_{15}^2 - T_{15},\, T_{12}T_{15} - T_{12},\, T_8T_{15} - T_8,\\
T_6T_{15} - T_{12},\, T_8T_{12} - T_{15},\, T_6T_9 + T_{15} - 1,\\
T_9^2T_{13} - T_6^2 + T_{12}^2,\, T_8^2T_{13} - T_{12}^2
\end{array}\right)\\
&\subseteq\; \mathbb{C}^{15}.
\end{aligned}
$$

Compute $\mathrm{Aut}(X)$

Example (continued)

For the Mori dream space X from before:

$$\mathrm{Aut}(X) \cong V\left(\begin{array}{l} T_{11},\, T_{10},\, T_7,\, T_5,\, T_9 T_{15}, \\ T_2 - T_9,\, T_1 - T_8,\, T_4 - T_6 + T_{12},\, T_{14} + T_{15} - 1, \\ T_3 - T_8 - T_9,\, T_{15}^2 - T_{15},\, T_{12} T_{15} - T_{12},\, T_8 T_{15} - T_8, \\ T_6 T_{15} - T_{12},\, T_8 T_{12} - T_{15},\, T_6 T_9 + T_{15} - 1, \\ T_9^2 T_{13} - T_6^2 + T_{12}^2,\, T_8^2 T_{13} - T_{12}^2 \end{array}\right) \subseteq \mathbb{C}^{15}.$$

Moreover:

- $\dim(\mathrm{Aut}(X)) = 1$ and $\left[\mathrm{Aut}(X) : \mathrm{Aut}(X)^0\right] = 2$.
- $\mathrm{Aut}(X) \cong \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{C}^*$.

Implementation in Singular

Software: for the free computer-algebra system Singular (Decker/Greuel/Pfister/Schönemann, 2015), we produce a library *autmds.lib*. Currently can compute $\mathrm{Aut}_K(R)$.

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Ingredients:

- Linear Algebra,
- Lattice points,
- Polyhedral cones,
- Hilbert bases,
- Gröbner bases.

Application: Automorphisms of cubic surfaces

Singular cubic surfaces with at most ADE singularities:

- $\mathrm{Aut}(X)$ for parameter-free cases: Sakamaki, *Trans. Amer. Math. Soc., 2010*.
- all Cox rings: Derenthal/Hausen/Heim/–/Laface, *J. Algebra, 2015*.

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Theorem

Let X be a singular cubic surface with ADE singularities. with general parameters. Then:

singularity type	$\mathrm{Aut}(X)$
A_3	$\mathbb{Z}/2\mathbb{Z}$
A_2A_1	$\{1\}$
$2A_2$	$\mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{C}^*$
$3A_1$	S_3
A_2	$\{1\}$
$2A_1$	$\mathbb{Z}/2\mathbb{Z}$
A_1	$\{1\}$